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Quantization of the moment map of coupled harmonic oscillators

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Abstract. We quantize the components of the momentum map of torus actions on some symplectic manifolds. These components may be understood as the Hamiltonians of commuting harmonic oscillators. With suitable choice of phase spaces this yields an interesting version of the Borel–Weil–Bott theorem interpreted as a result in geometric quantization. The present approach sheds some light on this phenomenon, which has been widely discussed from somewhat different points of view. In this paper we treat in full detail only a model example—the group $SU(3)$ because it contains almost all elements of the general case of compact Lie groups.

In almost all classical physically significant problems the phase spaces of Hamiltonian mechanics are cotangent bundles of manifolds with canonical symplectic structure [1–4]. The other source of symplectic manifolds in science is algebraic geometry which gives a lot of deep information about (compact) algebraic manifolds—the symplectic structure there is given by the Kahler form. The connection between these has been exploited over the last two centuries through the procedure which is now called reduction. Starting with the work of Jacobi (and implicitly by Kepler, Newton and Euler before him) this method has produced the most beautiful solutions of physical problems in terms of algebro-geometric entities as θ-functions etc.

One of the purposes of the present paper is to discuss the process of reduction from non-compact to compact (algebraic) phase spaces as a method for quantization, which we think is the basic fact of the geometric quantization scheme (see also [5] for a very interesting discussion of this subject). Compact complex (specially algebraic) manifolds involve a lot of discrete characteristics starting from continuous background. A further purpose of the paper is to show how naturally these spaces appear in classical mechanics, and how the quantum-mechanical picture arises when geometric quantization is applied to them. We are treating here the simple case of a phase space with a free symplectic action of a torus. The corresponding momentum map is a collection of commuting (in the Poisson sense) Hamiltonians (conservation laws). However simple this picture might be, a special case of it is the whole representation theory of semisimple Lie groups and the theory of universal spaces for vector bundles. These arise in our context as simultaneous quantization of a collection of coupled harmonic oscillators.

First we introduce some convenient notation for our case:

$$(\xi,\eta)=(\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3)\subset\mathbb{C}^3\times\mathbb{C}^3=\mathbb{C}^6$$
$$M=\{(\xi,\eta)\in\mathbb{C}^6\backslash\{0\}:\xi\eta=\xi_1\eta_1+\xi_2\eta_2+\xi_3\eta_3=0\} \qquad (1)$$

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$$\Omega = \frac{\mathrm{i}}{2}(\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi} + \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta})_{|M}. \tag{2}$$

Further we denote by K the torus $U(1) \times U(1)$, i.e.

$$K = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2; |\lambda_1| = |\lambda_2| = 1\}.$$

There is a symplectic action of K on (M, Ω)

$$\Phi : K \times M \to M$$

defined by

$$\Phi((\lambda_1, \lambda_2), (\xi, \eta)) = (\lambda_1 \xi, \lambda_2 \eta). \tag{3}$$

This action is in fact the flow generated on M by the harmonic oscillator Hamiltonians

$$J_1(\xi, \eta) = \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \xi_3\bar{\xi}_3 = |\xi|^2$$

and

$$J_2(\xi, \eta) = \eta_1\bar{\eta}_1 + \eta_2\bar{\eta}_2 + \eta_3\bar{\eta}_3 = |\eta|^2$$

which define, respectively

$$(\xi, \eta) \to (\lambda_1 \xi, \eta) = (\mathrm{e}^{\mathrm{i}t}\xi, \eta) \qquad \text{and } (\xi, \eta) \to (\xi, \lambda_2 \eta) = (\xi, \mathrm{e}^{\mathrm{i}s}\eta).$$

A different way to state the above is to say that the map:

$$J : M \to k^* \cong \mathbb{R}^2 \qquad J(\xi, \eta) = (J_1(\xi, \eta), J_2(\xi, \eta)) \tag{4}$$

is the momentum mapping of the action (3) (here k^* is the Lie coalgebra of K). We note specially that the action (3) of K is *free* on the symplectic submanifold $\hat{M} \subset M$, where

$$\hat{M} = \{(\xi, \eta) \in M \colon |\xi| > 0, |\eta| > 0\} \tag{5}$$

while on the symplectic submanifolds

$$M_1 = \{(\xi, \eta) \in M \colon |\eta| = 0\} \cong \mathbb{C}^3 \backslash \{0\}$$
$$M_2 = \{(\xi, \eta) \in M \colon |\xi| = 0\} \cong \mathbb{C}^3 \backslash \{0\}$$

it degenerates to obvious symplectic free $U(1)$ actions. Obviously M is the disjoint union of M_1, M_2 and $\hat{M}$. On the symplectic manifolds M_1 and M_2 we have two regular harmonic oscillators. In the spirt of Hertz' force-free mechanics their effective coupling is turned on in $\hat{M}$ via the constraint.

Now we are going to 'geometrically' quantize the momentum mapping $J \colon M \to k^*$. In general, the Marsden–Weinstein reduction theorem [6] gives us, for every regular value $\mu \in k^*$ with isotropy subgroup K_μ, an orbit manifold

$$\mathbb{O}_\mu = J^{-1}(\mu)/K_\mu$$

which is symplectic with symplectic form Ω_μ determined by the formula

$$\iota_\mu^*\Omega = \pi_\mu^*\Omega_\mu. \tag{6}$$

where $\iota_\mu : J^{-1}(\mu) \to M$ is inclusion, and $\pi_\mu : J^{-1}(\mu) \to \mathbb{O}_\mu$ is the canonical projection defined by (6).

Our Lie group acting on the symplectic manifold M is the (abelian) torus K and obviously we always have $K_\mu = K$. Now we are going to identify the orbit manifold $\mathbb{O}_\mu$ as a projective manifold (the final result for all μ is lemma 2, below). For generic μ the manifold $\mathbb{O}_\mu$ will be (isomorphic to) a fixed complex manifold and the *dependence on μ will be displayed by the symplectic form* Ω_μ (see lemma 3).

As a general reference for all the algebraic and complex geometry used further on, we propose [7]. We denote by $\mathbb{P}^n$ the complex n-dimensional projective space. If $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_n)$ are coordinates in $\mathbb{C}^{n+1}\backslash\{0\}$, then we shall use $\zeta = [\zeta_0 : \zeta_1 : \ldots : \zeta_n]$ to denote the projective coordinates on $\mathbb{P}^n$. Thus we have a well-defined map

$$h : \mathbb{C}^{n+1} \to \mathbb{P}^n, h(\zeta_0, \zeta_1, \ldots, \zeta_n) = [\zeta_0 : \zeta_1 : \ldots : \zeta_n]. \tag{7}$$

The restriction of the map h to any sphere of radius R

$$\mathbb{S}_R^{2n+1} = \{\zeta \in \mathbb{C}^{n+1} : |\zeta|^2 = R^2\}$$

is called the *Hopf map*, and will be denoted also by h.

We recall that the Hopf map coincides with the canonical projection associated to the $U(1)$ action on $\mathbb{S}^{2n+1}$ defined by

$$(\lambda, \zeta) \to \lambda\zeta \qquad \lambda \in U(1) \qquad \zeta \in \mathbb{S}_R^{2n+1} \qquad \text{for any } R > 0$$

(as in formula (4)). One can say that a Hopf map is the factorization by the flow of a harmonic oscillator of an isoenergetic surface.

We denote by $\mathbb{F}$ the flag manifold

$$\mathbb{F} = \{([\xi], [\eta]) \in \mathbb{P}^2 \times \mathbb{P}^2 : \xi\eta = 0\}. \tag{8}$$

Obviously we have two projections p_1 and p_2 of $\mathbb{F}$ on the respective factors

$$\begin{array}{ccc} & \mathbb{F} \subset \mathbb{P}^2 \times \mathbb{P}^2. & \\ \swarrow & & \searrow \\ p_1 & & p_2 \\ \mathbb{P}^2 & & \mathbb{P}^2 \end{array} \tag{9}$$

Both p_1 and p_2 define $\mathbb{F}$ as a projective bundle with base $\mathbb{P}^2$ and fibre $\mathbb{P}^1$, e.g.

$$p_1^{-1}([\xi]) = \{[\eta] \in \mathbb{P}^2 : \xi\eta = 0\}$$

is a projective line.

We denote by α the Fubini–Study Kahler form on $\mathbb{P}^2$ given in projective coördinates $[\zeta] = [\zeta_0 : \zeta_1 : \zeta_2]$ by the formula

$$\alpha(\zeta) = \frac{\mathrm{i}}{2}\frac{|\zeta|^2 \sum \mathrm{d}\zeta_a \wedge \mathrm{d}\bar{\zeta}_a - \sum \bar{\zeta}_a\zeta_b\,\mathrm{d}\zeta_a \wedge \mathrm{d}\bar{\zeta}_b}{|\zeta|^4}. \tag{10}$$

Besides (the cohomology class of), α is the positive generator of the group $H^2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$, and (the cohomology class of) the form α^2 generates $H^4(\mathbb{P}^2, \mathbb{Z})$. We denote

$$\omega_1 = \iota^*\alpha_1 \qquad \omega_2 = \iota^*\alpha_2. \tag{11}$$

By functionality and the Lefshetz hyperplane section theorem (see [7]) (the cohomology classes of) ω_1, ω_2 generate $H^2(\mathbb{F}, \mathbb{Z})$.

Lemma 1. The Chern classes of the complex manifold $\mathbb{F}$ are as follows:

$$c_1(\mathbb{F}) = 2(\omega_1 + \omega_2)$$
$$c_2(\mathbb{F}) = \omega_1^2 + 5\omega_1\omega_2 + \omega_2^2$$
$$c_3(\mathbb{F}) = 3(\omega_1^2\omega_2 + \omega_1\omega_2^2).$$

Proof. By definition as a submanifold of $\mathbb{P}^2 \times \mathbb{P}^2$ (see (8)), the manifold $\mathbb{F}$ is a divisor of the line bundle $V \to \mathbb{P}^2 \times \mathbb{P}^2$, for which

$$c_1(V) = \alpha_1 + \alpha_2$$

whence by (11) and functionality of Chern classes

$$c_1(V_{|\mathbb{F}}) = \omega_1 + \omega_2. \tag{12}$$

Also by (11), and functionality

$$c(\mathbb{T}(\mathbb{P}^2 \times \mathbb{P}^2)_{|\mathbb{F}}) = 1 + 3(\omega_1 + \omega_2) + 3(\omega_1^2 + 3\omega_1\omega_2 + \omega_2^2) + 9(\omega_1^2\omega_2 + \omega_1\omega_2^2).$$

The tangent bundle $\mathbb{T}(\mathbb{F})$ is obviously included in the following exact sequence

$$0 \to \mathbb{T}(\mathbb{F}) \to \mathbb{T}(\mathbb{P}^2 \times \mathbb{P}^2)_{|\mathbb{F}} \to V_{|\mathbb{F}} \to 0. \tag{13}$$

A standard computation with the Whitney sum formula

$$(1 + c_1(\mathbb{F}) + c_2(\mathbb{F}) + c_3(\mathbb{F}))(1 + \omega_1 + \omega_2) = c(\mathbb{T}(\mathbb{P}^2 \times \mathbb{P}^2)_{|\mathbb{F}})$$

yields directly the result of the lemma.

Lemma 2. Let $\mu = (\mu_1, \mu_2) \in k^*$. Then the orbit manifold $\mathbb{O}_\mu = J^{-1}(\mu)/K$ given by the reduction theorem is

(i) $\mathbb{O}_\mu = \mathbb{F}$ iff $\mu_1 > 0$ and $\mu_2 > 0$
(ii) $\mathbb{O}_\mu = \mathbb{P}^2$ iff $\mu_1 > 0$ and $\mu_2 = 0$
(iii) $\mathbb{P}_\mu = \mathbb{P}^2$ iff $\mu_1 = 0$ and $\mu_2 > 0$

and $\mathbb{O}_\mu = \emptyset$ otherwise.

Proof. In all cases above the canonical projection $\pi_\mu : J^{-1}(\mu) \to \mathbb{O}_\mu$ is constructed by using Hopf maps (see (7))

$$h : \mathbb{S}^5_R \to \mathbb{P}^2 \tag{14}$$

where the radius R is determined by μ (see below).

First we treat case (i). Let $\mu_1, \mu_2 > 0$. The isoenergetic surface $J^{-1}(\mu)$ is defined by the equation $\xi\eta = 0$, as a submanifold of

$$\{(\xi, \eta) \in M : |\xi|^2 = \mu_1; |\eta|^2 = \mu_2\} \cong S^5 \times S^5. \tag{15}$$

The action (3) is the restriction of the direct product of the two Hopf $U(1)$ actions from the manifold (15) to the submanifold $\hat{M} \cap S^5 \times S^5 = J^{-1}(\mu)$. Thus we have the direct product of two Hopf maps $h_1 \times h_2$ defining the canonical projection

$$\pi_\mu : J^{-1}(\mu) \to \mathbb{O}_\mu = J^{-1}(\mu)/K = \mathbb{F}$$

of the action (3), as described by the following diagram.

$$\begin{array}{ccc} \mathbb{S}^5 \times \mathbb{S}^5 & \xrightarrow{h_1 \times h_2} & \mathbb{P}^2 \times \mathbb{P}^2; \ h_1 \times h_2(\xi, \eta) = (\xi, \eta) \\ \cup & & \cup \\ J^{-1}(\mu) & \xrightarrow{\pi_\mu} & J^{-1}(\mu)/K = \mathbb{O}_\mu = \mathbb{F}. \end{array} \tag{16}$$

Comparing formula (8), (15), and (16) we obtain the proof of case (i).

To prove (ii) and (iii) we have only to notice that for example, in case (ii), the canonical projection

$$\pi_\mu : \{\xi \in M_1 : |\xi|^2 = \mu_1\} \to \mathbb{P}^2$$

is just the Hopf map (see (6) and (14)).

Remark. There is another way to describe the manifold $\hat{M}$ and its submanifolds $J^{-1}(\mu)$ for generic μ (case (i) of lemma 2).

Proposition. If $\mu_1 > 0$, $\mu_2 > 0$, then $J^{-1}(\mu)$ is diffeomorphic to $SU(3)$.

Proof. Let $F : \hat{M} \to \hat{M}$ be defined by

$$F(\xi, \eta) = \left(\frac{\xi}{|\xi|}, \frac{\eta}{|\eta|}\right).$$

For every $\mu = (\mu_1, \mu_2)$, such that $\mu_1 > 0$, $\mu_2 > 0$, the restriction

$$F_\mu = F_{|J^{-1}(\mu)}$$

defines a diffeomorphism

$$J^{-1}(\mu) \cong J^{-1}(1, 1) = \{(\xi, \eta) \in \mathbb{C}^3 \times \mathbb{C}^3 : |\xi|^2 = |\eta|^2 = 1, \xi\eta = 0\}.$$

There is an obvious diffeomorphism $\Theta : J^{-1}(1, 1) \to SU(3)$.

$$\Theta(\xi, \eta) = \begin{Vmatrix} \xi_0 & \xi_1 & \xi_2 \\ \bar{\eta}_0 & \bar{\eta}_1 & \bar{\eta}_2 \\ \zeta_0 & \zeta_1 & \zeta_2 \end{Vmatrix} \qquad \zeta_i = \varepsilon_{ijk}\bar{\xi}_j\eta_k. \tag{17}$$

Thus we obtain the desired diffeomorphism

$$\Theta_\mu = \Theta \circ F_\mu : J^{-1}(\mu) \to SU(3) \tag{18}$$

which proves proposition 1.

For the following we have to describe explicitly the map ι_μ for generic μ, which purpose is probably best served by the following diagram

$$\begin{array}{ccc} \mathbb{S}^5 \times \mathbb{S}^5 & \xrightarrow{\iota \times \iota} & \mathbb{C}^6 \\ \cup & & \cup \\ SU(3) \cong J^{-1}(\mu) & \xrightarrow{\iota_\mu} & \hat{M} \end{array} \tag{19}$$

Lemma 3. The reduced symplectic form on $\mathbb{O}_\mu$ (determined by formula (6)) is given by

(i) $\Omega_\mu = \mu_1\omega_1 + \mu_2\omega_2$ on $\mathbb{F}$
(ii) $\Omega_\mu = \mu_1\alpha$ on $\mathbb{P}^2$
(iii) $\Omega_\mu = \mu_2\alpha$ on $\mathbb{P}^2$.

Proof. We shall use the notation and facts from the proof of the preceding lemma. We have in fact already described the maps π_μ, ι_μ ((16), (19)), determining the form Ω_μ (see (6)).

In order to prove lemma 3 we have to pull the Fubini–Study form (10) with a Hopf map (7) from $\mathbb{P}^n$ back to a sphere

$$\mathbb{S}_R^{2n+1} \stackrel{\iota}{\subset} \mathbb{C}^{n+1}.$$

We just have to put $|\zeta|^2 = R^2$ in formula (10) to find

$$h^*\alpha = R^{-2}\iota^*\Omega. \tag{20}$$

Thus using formulae (16), (19) and (20) we obtain case (i):

$$\pi_\mu^*(\Omega_\mu) = \pi_\mu^*(\mu_1\omega_1 + \mu_2\omega_2) = \iota_\mu^*(\Omega).$$

The degenerate cases (ii) and (iii) follow readily from (20) and the description of π_μ in these cases at the end of the proof of lemma 2. The proof is completed.

We now recall some relevant facts about geometric quantization of compact Kahler manifolds [3, 8–11]. Let X be a Kahler manifold with Kahler form θ. By definition a holomorphic line bundle $L \to X$ is a *quantum bundle iff*

$$c_1(L) = \theta - \tfrac{1}{2}c_1(X). \tag{21}$$

A symplectic Kahler manifold possesses the canonical *anti-holomorphic polarization.* For our purposes this means that the quantum states are exactly the holomorphic sections of the quantum bundle L, i.e. the *quantum Hilbert space* associated to the classical phase space (X, θ) is the Hilbert space $(H^0(X, L), \langle\cdot,\cdot\rangle)$ where

$$\langle\phi, \psi\rangle = \int_X g(\phi, \psi)\Omega_\theta \qquad \phi, \psi \in H^0(X, L).$$

Here

$$\Omega_\theta = \frac{(-1)^{n(n-1)/2}}{n!}\theta \wedge \theta \wedge \ldots \wedge \theta.$$

The Hermitian metric g on L is determined by the condition, that its associated exterior form h is the harmonic representative in the cohomology class (21).

Thus a symplectic (Kahler) manifold (X, θ) is *quantizable* iff the above construction defines a genuine non-empty Hilbert space, which imposes the following conditions:

(a) $\theta - \frac{1}{2}c_1(X) \in H^2(X, \mathbb{Z})$;
(b) The harmonic representative h is a *positive form.*

We recall that, the curvature of the Hermitian metric g on the bundle L satisfies

$$\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log g \simeq \theta - \tfrac{1}{2}c_1(X).$$

The space $H^0(X, L)$ with the scalar product given by g is the Hilbert space of quantum states associated to the symplectic manifold (X, θ). To a classical observable (i.e. a function f) on the phase space, there corresponds a quantum operator

$$\delta(f) \in \operatorname{End} H^0(X, L) \qquad \delta(f)s \equiv (-\mathrm{i}\nabla_{X_f} + f)s$$

where $s \in H^0(X, L)$, and the vector field X_f is defined by:

$$\iota(X_f)h = -\mathrm{d}f.$$

Theorem. To each pair of non-negative integers (m_1, m_2) there corresponds exactly one admissible (quantum) value of the 'observable' $\mu \in k^*$ (defined by formula (4)):

(i) $(\mu_1, \mu_2) = (m_1 + 1, m_2 + 1)$ iff $m_1 \geqslant 0$ and $m_2 \geqslant 0$
(ii) $(\mu_1, \mu_2) = (m_1 + \frac{3}{2}, 0)$ iff $m_1 \geqslant 0$ and $m_2 = 0$
(iii) $(\mu_1, \mu_2) = (0, m_2 + \frac{3}{2})$ iff $m_1 = 0$ and $m_2 \geqslant 0$.

The *multiplicity* h_μ of a given admissible $\mu \in k^*$, i.e. the dimension of the corresponding quantum Hilbert space H_μ is given by the formula

$$h_\mu = \dim H^0(\mathbb{O}_\mu, L_\mu) = \tfrac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2) \tag{22}$$

where m_1, m_2 are the numbers specified above.

Remark. One could notice that the multiplicities (22) are substantially different from the multiplicities which would occur if the two oscillators were not interacting.

Proof. Let $\mu = (\mu_1, \mu_2)$ be an admissible value. We treat first the generic case $\mu_1 > 0$, $\mu_2 > 0$. By formula (21), the differential form

$$c_1(L_\mu) = \Omega_\mu - \tfrac{1}{2}c_1(\mathbb{F}) = (\mu_1 - 1)\omega_1 + (\mu_2 - 1)\omega_2 \in H^2(\mathbb{F}, \mathbb{Z})$$

whence

$$\mu_1 - 1 = m_1 \qquad \mu_2 - 1 = m_2 \in \mathbb{Z} \qquad m_1 \geqslant 0, m_2 \geqslant 0.$$

This settles the generic case.

Of the degenerate cases (ii), (iii) we shall perform the computations in only one, say (ii).

When $\mu_2 = 0$, then $\mu_1 > 0$, and the classical phase space is the symplectic manifold M_1:

$$c_1(L_\mu) = \Omega_\mu - \tfrac{1}{2}c_1(\mathbb{P}^2) = (\mu_1 - \tfrac{3}{2})\alpha \in H^2(\mathbb{P}^2, \mathbb{Z})$$

$$\mu_1 - \tfrac{3}{2} = m_1 \in \mathbb{Z} \qquad m_1 \geqslant 0$$

whence

$$\mu_1 = m_1 + \tfrac{3}{2} \qquad m_1 = 0, 1, 2, \ldots.$$

The computation of the multiplicities h_μ (formula (22) is, of course, just the computation of the (well-known) dimensions of the irreducible representations of $SU(3)$, or via the Borel–Weil–Bott theorem [12], the computation of the dimensions of the corresponding cohomology spaces. For completeness of the exposition of our quantization procedure we include this computation applying the Riemann–Roch theorem.

We treat first the generic case when $\mu_1 > 0$, $\mu_2 > 0$. In the chain of equalities that follow, the first is a consequence of the positiveness of the bundle L_μ and the Kodaira vanishing theorem ([7], ch 1); the second equality is the Riemann–Roch theorem; the third equality is an application of lemma 1 above, the substitution

$$f = m_1\omega_1 + m_2\omega_2 = c_1(L_\mu)$$

and a straightforward computation. For the definition and properties of the Euler characteristic $\chi(\cdot)$ and the Chern character ch(.) of a holomorphic bundle and the Todd class td(.) of a manifold, we refer to [7].

$$\begin{aligned} h_\mu = \chi(L_\mu) &= \langle \mathrm{ch}(L_\mu)\mathrm{td}(\mathbb{F}), \mathbb{F}\rangle \mathrm{ch}(L_\mu)\mathrm{td}(\mathbb{F}) = \{1 + f + \tfrac{1}{2}f^2 + \tfrac{1}{6}f^3\} \\ &\quad \times \{1 + c_1(\mathbb{F}) + \tfrac{1}{12}(c_1^2(\mathbb{F}) + c_2(\mathbb{F})) + \tfrac{1}{24}c_1(\mathbb{F})c_2(\mathbb{F})\} \\ &= \tfrac{1}{24}c_1(\mathbb{F})c_2(\mathbb{F}) + \tfrac{1}{12}f(c_1^2(\mathbb{F}) + c_2(\mathbb{F})) + \tfrac{1}{2}f^2c_1(\mathbb{F}) + \tfrac{1}{6}f^3 \\ &= \tfrac{1}{2}(\omega_1^2\omega_2 + \omega_1\omega_2^2) + \tfrac{1}{12}[(13m_1 + 5m_2)\omega_1^2\omega_2 + (5m_1 + 13m_2)\omega_1\omega_2^2)] \\ &\quad + \tfrac{1}{2}[(m_1^2+2m_1m_2)\omega_1^2\omega_2+(2m_1m_2+m_2^2)\omega_1\omega_2^2] + \tfrac{1}{2}(m_1^2m_2\omega_1^2\omega_2+m_1m_2^2\omega_1\omega_2^2). \end{aligned}$$

Integrating the above expression over the manifold $\mathbb{F}$

$$\begin{aligned} \langle \mathrm{ch}(L_\mu)\mathrm{td}(\mathbb{F}), \mathbb{F}\rangle &= 1 + \tfrac{3}{2}(m_1 + m_2) + (m_1^2 + 4m_1m_2 + m_2^2) + \tfrac{1}{2}m_1m_2(m_1 + m_2) \\ &= \tfrac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2) \end{aligned} \tag{23}$$

we obtain the desired result.

To complete the proof of the theorem, one must treat the degenerate cases (ii), (iii), when $\mu_1 = 0$ or $\mu_2 = 0$, whence $\mathbb{O}_\mu \cong \mathbb{P}^2$, in a similar way. *Formula (23) still holds when μ_1 or μ_2 vanishes.* One can expect this, keeping in mind that the corresponding representations of $SU(3)$ are already contained in the cohomology of $\mathbb{F}$ (see the remark below). Our theorem is proved.

Remark (continued): The torus K identifies with a maximal torus of the group $SU(3)$ as represented in formula (17) by

$$(\lambda_1, \lambda_2) \to \left\| \begin{matrix} \lambda_1 & 0 & 0 \\ 0 & \bar{\lambda}_2 & 0 \\ 0 & 0 & \bar{\lambda}_1\lambda_2 \end{matrix} \right\|. \tag{24}$$

It is well known [12] that the flag manifold $\mathbb{F}$ may be represented as follows:

$$\mathbb{F} = SL(3, \mathbb{C})/\mathrm{B} \cong SU(3)/K \tag{25}$$

where B is a Borel subgroup of $SL(3, \mathbb{C})$ (e.g. the upper triangular unimodular matrices).

Using the diffeomorphisms Θ_μ defined in the previous Remark (formula (18)) we can spread the *left action* of the torus K on $SU(3)$ to all manifolds $J^{-1}(\mu)$ with $\mu_1, \mu_2 > 0$. Thus we have

$$\hat{M} = \bigcup_{\mu_1,\mu_2>0} J^{-1}(\mu) \tag{26}$$

which defines a free action of the torus K on the symplectic manifold $\hat{M}$ by acting separately on each summand of (26). A straightforward check shows that this action coincides with the action defined by formula (3). In our approach to the reduction where the value of the momentum map (Hamiltonian) determines the reduced symplectic form and the class of (admissible) symplectic forms determined by the quantization conditions coincides with the class of (Chern classes of) positive holomorphic line bundles on $\mathbb{F}$. The present remark makes it clear that our method of quantization gives a direct identification of this class with the class of positive weights in the Cartan algebra k (as it should by the Borel–Weil–Bott theorem). We should mention also that the result of our paper [1], where a classical dynamical system is quantized, could be interpreted as treatment of the group SO(4) in the present context. All compact semisimple Lie groups may be treated in a similar way.

We should remark that quantization of the flag manifolds has been treated recently by different approaches and for different reasons by several authors [13–15]. It is beyond the scope of the present paper to discuss the relations between these results.

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